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STRESS-FOCUSING EFFECT IN A UNIFORMLY HEATED TRANSVERSELY ISOTROPIC SPHERE

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Abstract—The ray theory is applied to the stress-focusing effects in a uniformly heated transversely isotropic solid sphere. The stress-focusing effect is the phenomenon that, under an instantaneous heating, stress waves reflected from the free surface of the sphere result in very high stresses at the center. Using the ray theory, the Fourier transformed solution of stress waves in the sphere is sorted into rays according to the ray path of multiply-reflected waves. The inverse transform of each ray gives rise to the exact solution of the transient response up to the arrival time of the next ray. The results reveal that stresses peak periodically at regular intervals and the stress peak at the center of the sphere depends on the anisotropy of the sphere.

INTRODUCTION

When brittle materials are subjected to a rapid change in temperature, such as thermal shock, substantial stresses develop. Resistance to the weakening of the fracture under these conditions is called thermal endurance, thermal stress resistance, or thermal shock resistance. The effect of thermal stresses on different kinds of materials depends not only on material characteristics but also on stress level, stress distribution in the body, and stress duration. Hence, it is very important to determine the actual stress in a brittle material under given conditions of heat transfer.

In this paper we solve exactly the problem of thermal shock in a transversely isotropic sphere. When a transversely isotropic sphere is subjected to a uniform temperature rise, a stress wave occurs at the surface the moment thermal impact is applied. The stress wave at the surface proceeds radially inward to the center of the sphere. The waves may accumulate at the center and give rise to very large stress magnitudes, even though the initial thermal stress is relatively small. This phenomenon is called the stress-focusing effect in Ho (1976).

As for the study of the stress-focusing effect in a sphere, Mann-Nachbar and Nachbar (1970) obtained the closed-form solution by using the Laplace transformation. The solution, however, is given in the form of a series expansion in the Laplace transformed space. The inverse Laplace transformation of the solution is very difficult. Therefore, they only discussed the behavior of the stress focusing at the center of the sphere mathematically. Recently Hata (1991) formed an exact solution to the problem of stress-focusing effects in a uniform heated solid sphere by applying the ray theory. The results show the interesting phenomenon that the stresses caused by the stress-focusing effects peak periodically at regular intervals. This paper analyses the effect of thermal stress waves in a transversely isotropic sphere using the same method. By using the ray theory, the Fourier transformed solution of stress waves in a sphere is sorted into rays according to the ray path of multiply-reflected waves. The inverse transform of each ray gives rise to the exact solution of the transient response up to the arrival time of the next ray.

The numerical results give clear indications of a stress-focusing effect in a transversely isotropic sphere. We also show how the behavior of stress at the center depends on the anisotropy of the sphere mathematically.

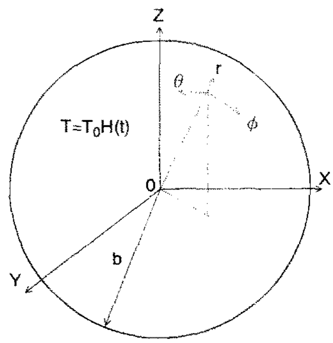

Fig. 1. Coordinate system and heat condition.

FORMULATION OF PROBLEM

A transversely isotropic, homogeneous, elastic solid sphere of radius b is subjected to a sudden uniform temperature rise as shown in Fig. 1. The stress–strain relations in a polar-symmetric transversely isotropic sphere are given from Hata and Atsumi (1969) by

$$\left.\begin{aligned} e_r &= S_r\sigma_r + S_{r\theta}\sigma_\phi + S_{r\theta}\sigma_\vartheta + \alpha_r T \\ e_\phi &= S_{r\theta}\sigma_r + S_\theta\sigma_\phi + S_{\phi\theta}\sigma_\vartheta + \alpha_\theta T \\ e_\theta &= S_{r\theta}\sigma_r + S_{\phi\theta}\sigma_\phi + S_\theta\sigma_\vartheta + \alpha_\theta T \end{aligned}\right\}, \tag{1}$$

where T is temperature and e_r, e_ϕ, e_θ are strain components of r, ϕ and θ directions, respectively. $\sigma_r, \sigma_\phi, \sigma_\vartheta$ are the principal stresses. α_r and α_θ are the coefficients of thermal expansion to the radial and tangential directions, respectively. The elastic compliance constants in eqn (1) can be written in terms of Young's moduli and Poisson's ratios as follows:

$$\left.\begin{aligned} S_r &= 1/E_r, \quad S_{r\theta} = -\nu_{r\theta}/E_r \\ S_\theta &= 1/E_\theta, \quad S_{\phi\theta} = -\nu_{\phi\theta}/E_\theta \end{aligned}\right\}, \tag{2}$$

where E_r and E_θ are Young's moduli in the radial and tangential planes, respectively. $\nu_{r\theta}$ is Poisson's ratio which characterizes the ratio of contraction in the θ-direction to extension in the r-direction, and $\nu_{\phi\theta}$ is the Poisson's ratio which characterizes the ratio of contraction in the θ-direction to the extension in the ϕ-direction. In this paper, $\sigma_\vartheta = \sigma_\phi$ and $e_\theta = e_\phi$ are satisfied because of the polar symmetry. Solving eqn (1) for σ_r and σ_ϑ, we get

$$\left.\begin{aligned} \sigma_r &= \frac{1}{(A+B)\omega_1}[(1-\nu_{\phi\theta})e_r + 2\nu_{r\theta}\omega_1 e_\theta - \xi_r^T] \\ \sigma_\vartheta &= \frac{1}{(A+B)}[\nu_{r\theta}e_r + e_\theta - \xi_\theta^T] \end{aligned}\right\}, \tag{3}$$

where

$$\left.\begin{aligned} \omega_1 &= E_\theta/E_r, \quad A = (S_{\phi\theta}S_r - S_{r\theta}^2)/S_r \\ B &= (S_\theta S_r - S_{r\theta}^2)/S_r \\ \xi_r^T &= (1-\nu_{\phi\theta})\alpha_r T + 2\nu_{r\theta}\omega_1\alpha_\theta T \\ \xi_\theta^T &= \nu_{r\theta}\alpha_r T + \alpha_\theta T \end{aligned}\right\}. \tag{4}$$

Under the condition of polar symmetry, the equation of motion is reduced to

$$\frac{\partial \sigma_r}{\partial r}+\frac{2(\sigma_r-\sigma_\vartheta)}{r}=\rho_0\frac{\partial^2 u}{\partial t^2}, \tag{5}$$

where ρ_0 is the mass per unit volume of a sphere and u is the radial displacement. The strain–displacement relations are

$$e_r=\partial u/\partial r,\quad e_\theta=u/r. \tag{6}$$

Substituting eqns (3) into (5) and using eqn (6), we have the displacement equation of equilibrium as

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-2\kappa\frac{u}{r^2}=\frac{1}{C_L^2}\frac{\partial^2 u}{\partial t^2}+\frac{1}{r^2}f^T(r,t), \tag{7}$$

where

$$\left.\begin{aligned} C_L&=\sqrt{\frac{(1-v_{\phi\theta})}{\rho_0(A+B)\omega_1}},\quad \kappa=\frac{(1-v_{r\theta})}{(1-v_{\phi\theta})}\omega_1\\ f^T(r,t)&=\frac{r^2}{(1-v_{\phi\theta})}\left[\frac{\partial\xi_r^T}{\partial r}+\frac{2}{r}(\xi_r^T-\omega_1\xi_\theta^T)\right]\end{aligned}\right\}, \tag{8}$$

where C_L is the dilatational wave speed.

For a transversely isotropic, homogeneous, elastic sphere, subject to uniform heating, the boundary condition is

$$\sigma_r(r,t)=0\quad\text{at}\quad r=b. \tag{9}$$

The sphere is at rest prior to time $t=0$ and the initial conditions are

$$u(r,t)=\partial u(r,t)/\partial t=0\quad\text{at}\quad t=0. \tag{10}$$

THERMOELASTIC SOLUTION

The temperature distribution is assumed to have the following form

$$T=T_0H(t), \tag{11}$$

where T_0 is a constant temperature and $H(t)$ is the Heaviside step function and t is time. Substituting eqn (11) into (3), the thermal stresses are given by

$$\left.\begin{aligned}\sigma_r^T&=\frac{T_0H(t)[(1-v_{\phi\theta}+2v_{r\theta}\omega_1)(l_r-\omega_1 l_\theta)-(1-\kappa)(1-v_{\phi\theta})l_r]}{(A+B)\omega_1(1-\kappa)(1-v_{\phi\theta})}\\ \sigma_\vartheta^T&=\frac{T_0H(t)[(v_{r\theta}+1)(l_r-\omega_1 l_\theta)-(1-\kappa)(1-v_{\phi\theta})l_\theta]}{(A+B)(1-\kappa)(1-v_{\phi\theta})}\end{aligned}\right\}, \tag{12}$$

where

$$l_r=(1-v_{\phi\theta})\alpha_r+2v_{r\theta}\omega_1\alpha_\theta,\quad l_\theta=v_{r\theta}\alpha_r+\alpha_\theta. \tag{13}$$

For an isotropic sphere eqns (12) reduce to $\sigma_r^T=\sigma_\vartheta^T=-(1+v_0)T_0\rho_0C_L^2\alpha_0H(t)/(1-v_0)$, where v_0 is Poisson's ratio and α_0 is the coefficient of thermal expansion in an isotropic sphere. The stresses of eqn (12) do not satisfy the boundary condition of eqn (9). In order to satisfy the boundary condition we must introduce the elastodynamic solution u^S such that the final solution $u=u^T+u^S$ satisfies all the boundary conditions of the problem. The corresponding boundary condition to u^S is that the radial displacement at $r = 0$ vanishes and the radial stress σ_r^S satisfies the condition

$$\sigma_r^S(r, t) = -\sigma_r^T \quad \text{at} \quad r = b. \tag{14}$$

For convenience, it is appropriate to introduce non-dimensional quantities which are given by

$$\left.\begin{aligned} &\zeta^i = u^i/b, \quad \tau = tC_L/b, \quad \rho = r/b \\ &\sigma_\rho^i = \sigma_r^i/(\alpha_r T_0 \rho_0 C_L^2) \\ &\sigma_\theta^i = \sigma_\vartheta^i/(\alpha_r T_0 \rho_0 C_L^2) = \sigma_\phi^i/(\alpha_r T_0 \rho_0 C_L^2), \quad (i = T, S) \end{aligned}\right\}. \tag{15}$$

The basic equation of the elastodynamic solution is derived from eqn (7) as

$$\frac{\partial^2 \zeta^S}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial \zeta^S}{\partial \rho} - 2\kappa \frac{\zeta^S}{\rho^2} = \frac{\partial^2 \zeta^S}{\partial \tau^2}. \tag{16}$$

We shall make use of the Fourier transform on τ, defined as

$$\left.\begin{aligned} \bar{\zeta}(\rho, \alpha) &= \int_0^\infty \zeta(\rho, \tau)\, e^{i\alpha\tau}\, d\tau \\ \zeta(\rho, \tau) &= \frac{1}{2\pi}\int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \bar{\zeta}(\rho, \alpha)\, e^{-i\alpha\tau}\, d\alpha \end{aligned}\right\}. \tag{17}$$

Applying the Fourier transform to eqn (16), the transformed solution which satisfies the boundary conditions is

$$\bar{\zeta}^S = \frac{-\bar{p}(\alpha)}{C_{12}+C_{22}}[g_\nu^{(1)}(\alpha\rho) + g_\nu^{(2)}(\alpha\rho)], \tag{18}$$

where

$$\left.\begin{aligned} \kappa &= (1-\nu_{r\theta})\omega_1/(1-\nu_{\phi\theta}) \\ \nu + (1/2) &= \sqrt{1+8\kappa}/2 \\ C_{12} &= \alpha g_{\nu-1}^{(1)}(\alpha b) + mb^{-1} g_\nu^{(1)}(\alpha b) \\ C_{22} &= \alpha g_{\nu-1}^{(2)}(\alpha b) + mb^{-1} g_\nu^{(2)}(\alpha b) \\ \bar{p}(\alpha) &= \bar{\sigma}_r^T(b, \alpha)/(\alpha_r T_0 \rho_0 C_L^2) \\ m &= 2\nu_{r\theta}\omega_1/(1-\nu_{\phi\theta}) - (\nu+1) \end{aligned}\right\}. \tag{19}$$

The functions $g_\nu^{(1)}(z)$ and $g_\nu^{(2)}(z)$ in eqns (18) and (19) are given by

$$\left.\begin{aligned} g_\nu^{(1)}(z) &= \sqrt{\pi/(2z)}H_{\nu+1/2}^{(1)}(z) \\ g_\nu^{(2)}(z) &= \sqrt{\pi/(2z)}H_{\nu+1/2}^{(2)}(z) \end{aligned}\right\}, \tag{20}$$

where $H_{\nu+1/2}^{(1)}(z)$ and $H_{\nu+1/2}^{(2)}(z)$ are the Hankel functions of the first and second kinds of order $\nu + 1/2$, respectively.

Here, we introduce a reflection coefficient R, which is defined as

$$R(\alpha) = -\frac{C_{12}}{C_{22}}. \tag{21}$$

The function R is the reflection coefficient for an outgoing spherical wave, which is reflected at the traction-free surface $\rho = 1$. Hence, we have from eqn (18) that

$$\bar{\zeta}^S = -\frac{\bar{p}(\alpha)}{C_{22}}(1+R+R^2+R^3+\cdots)[g_\nu^{(1)}(\alpha\rho)+g_\nu^{(2)}(\alpha\rho)] = \sum_{j=0}^{\infty} \bar{\Phi}_j(\rho,\alpha), \tag{22}$$

where

$$\left.\begin{aligned} \bar{\Phi}_0(\rho,\alpha) &= -\bar{p}(\alpha)g_\nu^{(2)}(\alpha\rho)/C_{22} \\ \bar{\Phi}_1(\rho,\alpha) &= -\bar{p}(\alpha)g_\nu^{(1)}(\alpha\rho)/C_{22} \\ \bar{\Phi}_2(\rho,\alpha) &= -R(\alpha)\left\{\frac{\bar{p}(\alpha)}{C_{22}}\right\}g_\nu^{(2)}(\alpha\rho) \\ \bar{\Phi}_3(\rho,\alpha) &= -R(\alpha)\left\{\frac{\bar{p}(\alpha)}{C_{22}}\right\}g_\nu^{(1)}(\alpha\rho) \\ \bar{\Phi}_j(\rho,\alpha) &= -R(\alpha)\bar{\Phi}_{j-2}(\rho,\alpha) \quad (j \geqslant 4) \end{aligned}\right\}. \tag{23}$$

The corresponding transformed stresses are given from eqn (3) by

$$\left.\begin{aligned} \bar{\sigma}_\rho^S &= \left[\frac{\partial\bar{\zeta}^S}{\partial\rho} + \frac{2\nu_{r\theta}\omega_1}{(1-\nu_{\phi\theta})}\frac{\bar{\zeta}^S}{\rho}\right] = \sum_{j=0}^{\infty}\left[\bar{\Phi}_{j,\rho} + \frac{2\nu_{r\theta}\omega_1}{(1-\nu_{\phi\theta})}\frac{\bar{\Phi}_j}{\rho}\right] \\ \bar{\sigma}_\theta^S &= \frac{\omega_1}{(1+\nu_{\phi\theta})}\left[\nu_{r\theta}\frac{\partial\bar{\zeta}^S}{\partial\rho} + \frac{\bar{\zeta}^S}{\rho}\right] = \frac{\omega_1}{(1-\nu_{\phi\theta})}\sum_{j=0}^{\infty}\left[\nu_{r\theta}\bar{\Phi}_{j,\rho} + \frac{\bar{\Phi}_j}{\rho}\right] \end{aligned}\right\}. \tag{24}$$

The inverse transform of each term represents the transient wave which is continuously reflected between the center and the surface. The complete solution is

$$\zeta^S(\rho,\tau) = \sum_{j=0}^{\infty}\frac{1}{2\pi}\int_{i\varepsilon_0-\infty}^{i\varepsilon_0+\infty}\bar{\Phi}_j(\rho,\alpha)\,e^{-i\alpha\tau}\,d\alpha. \tag{25}$$

TRANSIENT SOLUTION OF ANISOTROPIC SPHERE

As an explicit example, we consider the problem of the stress-focusing effect in a uniformly heated transversely isotropic sphere. The anisotropic property of the sphere is taken to be $\kappa = 3$ ($\nu = 2$) in eqn (19). From eqn (20) we have

$$\left.\begin{aligned} g_2^{(1)}(z) &= h_2^{(1)}(z) = -z^{-3}\{3z+i(3-z^2)\}\,e^{iz} \\ g_2^{(2)}(z) &= h_2^{(2)}(z) = -z^{-3}\{3z-i(3-z^2)\}\,e^{-iz} \end{aligned}\right\}, \tag{26}$$

where $h_2^{(1)}(z)$ and $h_2^{(2)}(z)$ are the second order spherical Hankel functions of the first and second kinds, respectively. Substituting eqns (23) into (25), the inverse Fourier transforms of $\bar{\Phi}_0(\rho,\alpha)$ and $\bar{\Phi}_1(\rho,\alpha)$ are given by

$$\left.\begin{aligned} \phi_0(\rho,\tau) &= -\frac{p}{\rho^3}\sum_{j=0}^{3}\frac{\{3\alpha_j\rho-\mathrm{i}(3-\alpha_j^2\rho^2)\}H(\tau+\rho-1)\,\mathrm{e}^{-\mathrm{i}\alpha_j(\tau+\rho-1)}}{\{4\alpha_j^3+3\mathrm{i}(m-1)\alpha_j^2+6m\alpha_j-3m\mathrm{i}\}} \\ \phi_1(\rho,\tau) &= -\frac{\rho}{\rho^3}\sum_{j=0}^{3}\frac{\{3\alpha_j\rho+\mathrm{i}(3-\alpha_j^2\rho^2)\}H(\tau-\rho-1)\,\mathrm{e}^{-\mathrm{i}\alpha_j(\tau-\rho-1)}}{\{4\alpha_j^3+3\mathrm{i}(m-1)\alpha_j^2+6m\alpha_j-3m\mathrm{i}\}} \\ p &= -\sigma_\rho^T \end{aligned}\right\}. \tag{27}$$

The poles α_j ($j = 1, 2, 3, 4$) in eqns (27) are

$$\alpha_1 = 0, \quad \alpha_2 = \gamma_1, \quad \alpha_3 = \gamma_2, \quad \alpha_4 = \gamma_3, \tag{28}$$

where

$$\left.\begin{aligned} \gamma_1 &= \mathrm{i}\{\sqrt[3]{A_0}-d/\sqrt[3]{A_0}-(m-1)/3\} \\ \gamma_2 &= \mathrm{i}\{\omega\sqrt[3]{A_0}-d/(\omega\sqrt[3]{A_0})-(m-1)/3\} \\ \gamma_3 &= \mathrm{i}\{\omega^2\sqrt[3]{A_0}-d/(\omega^2\sqrt[3]{A_0})-(m-1)/3\} \\ A_0 &= -l+\sqrt{l^2+d^3}, \quad \omega = (-1+\sqrt{3}\mathrm{i})/2 \\ l &= \{(m-1)/3\}^3+m(m-1)/2+3m/2 \\ d &= -\{(m-1)/3\}^2-m \end{aligned}\right\}. \tag{29}$$

It is seen from eqn (23) that, once the inversions of $\bar{\phi}_0$ and $\bar{\phi}_1$ are carried out, the higher order terms in the series can be obtained by a convolution. Denoting the inverse transform of $\bar{\phi}_j(\rho,\alpha)$ by $\phi_j(\rho,\tau)$, we find from eqn (23) that

$$\phi_j(\rho,\tau) = \int_0^\tau R(\xi)\phi_{j-2}(\rho,\tau-\xi)\,\mathrm{d}\xi \quad (j \geqslant 2), \tag{30}$$

where

$$R(\tau) = -\delta(\tau-2)+\mathrm{i}\sum_{j=1}^{2} q(\gamma_j)H(\tau-2). \tag{31}$$

In eqn (31) $\delta(\tau)$ is a delta function and the function $q(\gamma)$ is

$$q(\gamma) = -\frac{2\mathrm{i}\{\gamma^2(m-1)-3m\}\exp\{-\mathrm{i}\gamma(\tau-2)\}}{\{\gamma^3+\gamma^2\mathrm{i}(m-1)+3m\gamma-3\mathrm{i}m\}}. \tag{32}$$

Therefore, the elastodynamic stresses are given from the displacement potential $\phi_j(\rho,\tau)$ by

$$\left.\begin{aligned} \sigma_\rho^S(\rho,\tau) &= \sum_{j=0}^{\infty}[\phi_j(\rho,\tau)_{,\rho}+\{(m+3)/\rho\}\phi_j(\rho,\tau)] \\ \sigma_\theta^S(\rho,\tau) &= \sum_{j=0}^{\infty}\{(m+3)/2\}[\phi_j(\rho,\tau)_{,\rho}+(1/\nu_{r\theta}\rho)\phi_j(\rho,\tau)] \end{aligned}\right\}, \tag{33}$$

where $m = -3/7$ in the case of the anisotropic sphere of $\kappa = 3$ in eqn (19).

Finally, the complete solution of the problem is obtained by the summation of the solutions of the thermal and dynamic problems as follows:

$$\sigma_i(\rho,\tau) = \sigma_i^T(\rho,\tau)+\sigma_i^S(\rho,\tau), \quad (i = \rho,\theta), \tag{34}$$

where

$$\sigma_\rho(\rho,\tau) = \sigma_r(\rho,\tau)/(\alpha_r T_0 \rho_0 C_L^2), \quad \sigma_\theta(\rho,\tau) = \sigma_\vartheta(\rho,\tau)/(\alpha_r T_0 \rho_0 C_L^2). \tag{35}$$

NUMERICAL RESULTS AND DISCUSSION

When a transversely isotropic solid sphere is brought suddenly to a uniform temperature rise, stress-focusing effects occur. In order to analyse the phenomena, we performed numerical calculations for a transversely isotropic sphere with $\kappa = 3$, which was heated suddenly to a uniform temperature T_0 at $t = 0$. The numerical integrations were carried out using Simpson's formula. The results of the numerical evaluation of stress variation are illustrated in Figs 2–4. In these figures the non-dimensional variables t^* and σ_i^* are used, which are defined as

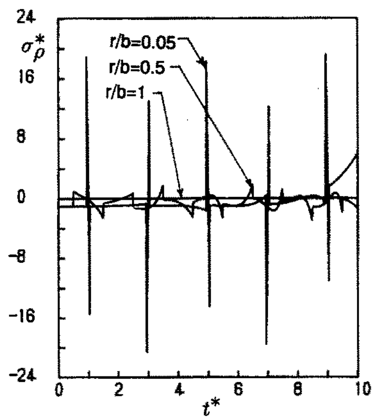

Fig. 2. Stress-focusing effect of the radial stress σ_ρ^* in a transversely isotropic sphere.

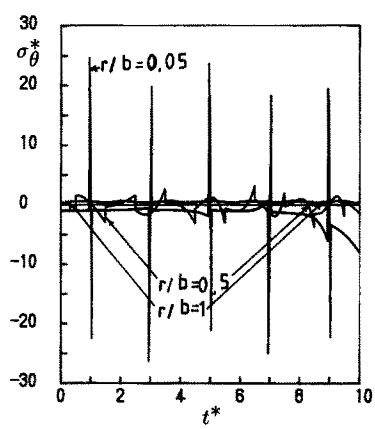

Fig. 3. Stress-focusing effect of the tangential stress σ_θ^* in a transversely isotropic sphere.

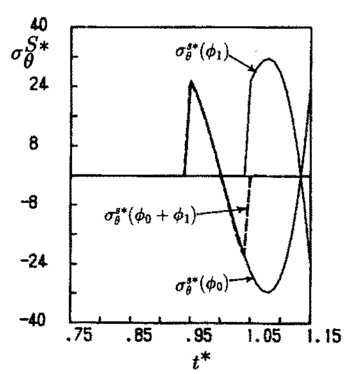

Fig. 4. Variation of σ_θ^{S*} as a function of individual rays and sum of the first two rays at $r/b = 0.05$.

$$t^* = C_L t/b, \quad \sigma_i^* = (\sigma_i/\sigma_\rho^T) \quad (i = \rho, \theta). \tag{36}$$

The behavior of radial stress σ_ρ^* as a function of time is illustrated in Fig. 2 at the location $r/b = 0.05$, 0.5 and 1. In Fig. 2 the stress σ_ρ^* is initially compressive because of the instantaneous heating. After this period, the waves are reflected from the surface and accumulate at the center. As the wave approaches the center, the maximum value of σ_ρ^* increases higher and higher as shown in Fig. 2. The peak stress σ_ρ^* at $r/b = 0.05$ peaks periodically at an interval of $t^* = 2$.

Figure 3 shows the variation in tangential stress σ_θ^* at different values of r/b as a function of time. The behavior of tangential stress σ_θ^* is similar to that of the radial stress. In the figures we also observe that the stress near the center peaks periodically.

Comparing these results with those in an isotropic sphere in Hata (1991), it is seen that the stress-focusing effects occur when stress waves are reflected from the surface of the sphere and accumulate radially toward the center. However, we find that the compressive stress peaks are observed in Figs 2 and 3 for a transversely isotropic sphere, but they are not observed in an isotropic sphere.

We should explain the reason why the stress peak at $\rho = 0.05$ changes from tension to compression at $t^* = 1$ in Fig. 3. The values of the first two rays of $\sigma_\theta^{S^*}$ at $\rho = 0.05$ are shown in Fig. 4. The value of the sum of the first two rays, $\sigma_\theta^{S^*}(\phi_1+\phi_2)$, of $\sigma_\theta^{S^*}$ is also shown. In Fig. 4 the first ray, $\sigma_\theta^{S^*}(\phi_0)$, arrives at $t^* = 0.95$ at $\rho = 0.05$, obtained from eqn (27), and diverges in time t^*. The second ray, $\sigma_\theta^{S^*}(\phi_1)$, arrives at $t^* = 1.05$ at $\rho = 0.05$, obtained from eqn (27), and also diverges in time t^*. However, we can observe that the sum of the first and second rays, $\sigma_\theta^{S^*}(\phi_0+\phi_1)$, of the stress $\sigma_\theta^{S^*}$ shows the positive peak at $t^* = 0.95$ and the stress peak changes from positive to negative sharply the moment the second ray arrives. After that, a pair of rays cancel each other and the sum is convergent. The same phenomena can be observed in other ray groups in Figs 2 and 3.

Here, we discuss the behavior of displacement and stresses at the center of the sphere precisely. Substitution of eqns (20) into (22) yields

$$\begin{aligned}\bar{\zeta}^S &= -\frac{\bar{p}(\alpha)}{C_{22}}(1+R+R^2+R^3+\cdots)[g_\nu^{(1)}(\alpha\rho)+g_\nu^{(2)}(\alpha\rho)] \\ &= -\frac{\bar{p}(\alpha)}{C_{22}}(1+R+R^2+R^3+\cdots)\sqrt{\frac{\pi}{2\alpha\rho}}\,J_{\nu+1/2}(\alpha\rho),\end{aligned} \tag{37}$$

where $J_{\nu+1/2}(z)$ is the $\nu+1/2$ order Bessel function of the first kind.

The Bessel function $J_{\nu+1/2}(z)$ near $z = 0$ takes the form

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu, \tag{38}$$

where $\Gamma(z)$ is the Gamma function. Substituting eqn (38) into (37), we find that the non-dimensional transformed displacement is

$$\bar{\zeta}^S = O(\rho^\nu). \tag{39}$$

Since ν is positive because $\kappa > 0$ in eqn (19), the transformed displacement $\bar{\zeta}^S$ becomes zero as ρ approaches zero. Therefore the radial displacement at $\rho = 0$ vanishes, and the boundary condition is satisfied.

Next, we discuss the behavior of stresses at the center of the sphere. From eqn (24) the transformed radial stress is given by

$$\bar{\sigma}_\rho^S = (1+R+R^2+R^3+\cdots)\left[(\bar{\phi}_0+\bar{\phi}_1)_{,\rho}+\frac{2v_{r\theta}\omega_1}{(1-v_{\phi\theta})}\frac{(\bar{\phi}_0+\bar{\phi}_1)}{\rho}\right]$$

$$= -\left(\frac{\bar{p}(\alpha)}{C_{22}}\right)\sqrt{\frac{\pi}{2}}(1+R+R^2+R^3+\cdots)\alpha\left[\left\{v+\frac{2v_{r\theta}\omega_1}{(1-v_{\phi\theta})}\right\}\frac{J_{v+1/2}(\alpha\rho)}{(\alpha\rho)^{3/2}}-\frac{J_{v+3/2}(\alpha\rho)}{(\alpha\rho)^{1/2}}\right]. \quad (40)$$

Substituting eqn (38) into (40) the radial stress may be denoted by

$$\bar{\sigma}_\rho^S = O(\rho^{v-1}). \quad (41)$$

Here, we encounter these singularities of distribution of stresses in a body with curvilinear anisotropy which, in the case of the existence of an axis of anisotropy, was noted in Lekhnitskii (1963). For $v < 1$ the center of the sphere must be excluded from consideration by surrounding it with a spherical surface of small radius. When v is not less than one, the radial stress $\bar{\sigma}_\rho^S$ takes two different values when ρ is equal to zero as follows:

$$\left.\begin{array}{lll}\bar{\sigma}_\rho^S = 0 & \text{when} \quad v > 1 & \text{(anisotropic case)}\\ \bar{\sigma}_\rho^S = O(1) & \text{when} \quad v = 1 & \text{(isotropic case)}\end{array}\right\}. \quad (42)$$

The foregoing discussion is also true in the case of the tangential stress. Therefore, in the case of a transversely isotropic sphere with $v = 2$ which is treated in the paper, we observe that the stresses near the center increase more and more as the wave approaches the center, as shown in Figs 2–4. These stresses, however, must vanish at the center of the sphere from eqn (42). It is concluded that the stress-focusing effect may occur near the center of the sphere, and not at the center of the sphere.

Here, we discuss the singular behaviors of both displacement and stresses in the neighborhood of the center of a transversely isotropic sphere with the properties of $\kappa = 3$ ($v = 2$). The stress motion generated in an initially undisturbed sphere by the application of a uniform temperature may be analysed from eqns (27)–(33). We find that the phase of the first incoming wave denoted by the function ϕ_0 is $\alpha_j(\tau+\rho-1)$, while the phase of the first outgoing wave denoted by the function ϕ_1 is $\alpha_j(\tau-\rho-1)$. These two waves interfere with each other in the neighborhood of $\rho = 0$ at time $\tau = 1$. An inspection of eqn (27) suggests that the first incoming wave denoted by ϕ_0 rises sharply in the order of ρ^{-3}, while the first outgoing wave denoted by ϕ_1 falls sharply in the order of ρ^{-3}. Therefore, the infinity of center displacement in the sphere behaves as a Dirac delta function. From eqn (30) it can be seen that the other singularities also behave like Dirac delta functions. The foregoing discussion is true for the case of stress distributions. Substituting eqns (27) and (30) into (33), we find the order of singularity of stress distribution is $O(\rho^{-4})$.

CONCLUSION

In this paper, applying the ray theory to the problem of stress-focusing effect in a uniformly heated transversely isotropic sphere, we show that the complicated Fourier transformed solutions of the problem lead to simple solutions, which are appropriate to the application of the inversion of Fourier transform. We also point out that the behavior of stress-focusing effect at the center of the transversely isotropic sphere depends on the anisotropy of the sphere.

Numerical results show that the peaks of stresses due to rapid heating become higher and higher as the waves propagate toward the center of a sphere and the peaks of compressive stress are observed only in a transversely isotropic sphere.

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